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# Hybrid Algebraic Iterative Reconstruction and Physics-Informed Neural Networks for Solving Inverse Problems

**H.K. Al-Mahdawi** [1,*]

[1] Electronic and Computer Centre, Diyala University, 32001 Diyala, Iraq.

## ABSTRACT

Algebraic Iterative Reconstruction (AIR) methods, including ART, SIRT, and SART, are effective techniques for addressing inverse problems in tomography, medical imaging, and industrial inspection. But their ability to reconstruct things gets worse in systems that are poorly conditioned, measurements that are noisy, and projection data that is sparse. Physics-Informed Neural Networks (PINNs), on the other hand, include governing differential equations in the learning framework. This makes it possible to reconstruct solutions even when the data is limited or only partially corrupted. This article suggests a combination of AIR and PINN that combines neural training based on PDEs with iterative algebraic solvers. The AIR step makes sure that the algebraic updates are stable, and the PINN makes sure that they are consistent with the physics that underlies them. Compared to AIR or PINN methods used on their own, the numerical results show better convergence, less noise sensitivity, and higher reconstruction accuracy.

**Keywords** Algebraic Iterative Reconstruction (AIR), Algebraic Reconstruction Technique (ART), Simultaneous Iterative Reconstruction Technique (SIRT), Simultaneous Algebraic Reconstruction Technique (SART).

***Corresponding Author**

***H.K. Al-Mahdawi***
***Department of Computer Science, College of Education for Pure Science,***
*University of Diyala, Baqubah City, Diyala Governorate,3200, Iraq.*
***Email**: hssnkd@gmail.com*



## 1. INTRODUCTION

Inverse problems are present in numerous scientific and engineering domains, such as computed tomography, seismic imaging, heat conduction analysis, and nondestructive material assessment. These issues are typically expressed as linear or nonlinear systems represented by $Ax = b$, where A is the system matrix that comes from measurement operators, x is the unknown field or image, and b is the data that was seen. If the projection data is noisy, missing, or only has a small range of angles, this system becomes ill-conditioned or underdetermined. This makes direct inversion unstable and very sensitive to mistakes.

Algebraic Iterative Reconstruction (AIR) methods offer a traditional methodology for resolving inverse systems via iterative correction. The Algebraic Reconstruction Technique (ART), which Kaczmarz first used for linear systems and later for tomography [1], the Simultaneous Iterative Reconstruction Technique (SIRT) [2], and the Simultaneous Algebraic Reconstruction Technique (SART) [3] are all well-known AIR methods. These algorithms gradually improve the solution by projecting residuals back into the solution space. This makes convergence better when analytical or closed-form solutions don't work. Even though AIR methods work well, they are very sensitive to measurement noise, don't have built-in physical priors, and can make unstable reconstructions in settings that aren't well-posed [4].

To mitigate these constraints, Physics-Informed Neural Networks (PINNs) have recently emerged as a robust alternative for addressing PDE-governed inverse problems by integrating the governing differential equations, boundary conditions, and physical constraints directly into the learning framework [5]. PINNs reconstruct fields that are consistent with physical laws even when data is sparse or partially corrupted. They do this by minimizing a loss function that includes data fidelity, PDE residuals, and boundary enforcement. This ability has been shown to work in a number of PDE-based systems, such as diffusion, elasticity, wave propagation, electromagnetic fields, and quantum systems [6].

But when data constraints are the most important thing, PINNs alone may have trouble with algebraic consistency, and when it comes to large-scale reconstructions, purely data-driven learning may take a long time to converge. It makes sense, then, to think about a hybrid framework in which AIR methods give algebraic stability and PINNs use PDE-constrained learning to impose physical regularization. When these methods are used together, they create a reconstruction process that is both algebraically consistent and physically meaningful. This makes ill-posed inverse problems more accurate and faster to solve.

This paper proposes a hybrid AIR–PINN reconstruction method in which iterative algebraic solvers create projection-consistent updates and PINNs improve the estimates by enforcing PDE structure. The goal of this integration is to make the system more stable, less sensitive to noise, and able to create reconstructions that are consistent with both measurement data and the laws of physics.

## 2. Mathematical Formulation

Inverse reconstruction problems are commonly represented by a finite-dimensional linear system derived from measurement operators or projection models:

$$Ax = b \tag{1}$$

where

- $A \in R^{m \times n}$ is the system or projection matrix,
- $x \in R^{n}$ is the unknown state (image, field, or material distribution),
- $b \in R^{m}$ is the measured data.

When m< n or when measurements are noisy, (1) becomes underdetermined or ill-posed, and no unique or stable solution exists without additional structural information [4]. In many physical systems, the unknown quantity *u(x)* also satisfies a governing differential equation:

$$N[u(x)] = 0, x \in \Omega, \tag{2}$$

with boundary or initial conditions:

$$B[u(x)] = g(x), x \in \partial\Omega, \tag{3}$$

where Ω denotes the computational domain and ∂Ω its boundary.

Algebraic reconstruction seeks a solution of (1) by iterative refinement. A general AIR update can be expressed as:

$$x^{k+1} = x^{k} + \alpha A \ (b - Ax^{k}), \tag{4}$$

where α is a relaxation or step-size parameter controlling update stability. Specific AIR instances include:

- **ART (Kaczmarz Method)** — sequential projections [1]:

$$x^{k+1} = x^{k} + \frac{b_i - \langle a_i, x^{k} \rangle}{\|a_i\|^2} a_i^{T} \tag{5}$$

- **SIRT** — simultaneous update averaging all projections [2]:

$$x^{k+1} = x^{k} + CA \ (b - Ax^{k}), \tag{6}$$

where *C* is a diagonal normalization matrix.

- **SART** — weighted projection correction for fast convergence [3]:

$$x^{k+1} = x^{k} + WA \ (b - Ax^{k}), \tag{7}$$

with *W* representing projection-dependent weights.

These schemes improve algebraic consistency with measured data; however, they cannot guarantee physical

Let $u_\theta(x)$ be a neural network approximation of the unknown state parameterized by weights *θ*. The PINN framework enforces physics by minimizing a composite loss:

$$L(\theta) = \| u_\theta - u_{data} \|^2 + \lambda_f \| N[u_\theta] \|^2 + \lambda_b \| B[u_\theta] - g \|^2. \tag{8}$$

Minimizing (8) yields a solution consistent with (2)–(3) and stabilizes reconstruction even with limited data [5], [6].

2.3 Coupled Algebraic–Physics Reconstruction Problem

We seek a solution that satisfies both algebraic observations and physical laws:

$$Find \ \ u(x) \ \ such \ that \ \ Au = b \ \ and \ \ N[u] = 0, \ B[u] = g \tag{9}$$

this leads to the hybrid problem:

$$u^{*} = \arg\min \|Au - b\|^{2} + \lambda_{f} \|N[u]\|^{2} + \lambda_{b} \|u - g\|_{\partial\Omega}^{2}$$
(10)

where $u$ may represent temperature, wave amplitude, pressure field, electron density, or attenuation coefficients depending on the application domain.Equation (10) provides the foundation for constructing a hybrid AIR–PINN reconstruction method, where AIR enforces algebraic consistency and PINNs impose PDE–physics regularity.

## 3. Hybrid AIR–PINN Algorithm (Proposed Method)

The proposed reconstruction method alternates between an **Algebraic Iterative Reconstruction (AIR)** update ensuring data-consistency and a **Physics-Informed Neural Network (PINN)** correction imposing PDE-governed physical structure. Let $u^{k}$ denote the reconstruction estimate at iteration *k*.

Starting from the linear observation model $Au=b$, the algebraic correction step is defined as:

$$x^{k+\frac{1}{2}} = x^{k} + \alpha A \;\; (b - Ax^{k}), \tag{11}$$

where $\alpha>0$ is a relaxation parameter. For specific AIR schemes:

- ART / Kaczmarz:

$$x^{k+\frac{1}{2}} = x^{k} + \frac{b_{i} - \langle a_{i}, x^{k} \rangle}{\|a_{i}\|^{2}} a_{i}^{T} \tag{12}$$

- SIRT:

$$x^{k+\frac{1}{2}} = x^{k} + CA \;\; (b - Ax^{k}), \tag{13}$$

- SART:

$$x^{k+\frac{1}{2}} = x^{k} + WA \;\; (b - Ax^{k}), \tag{14}$$

where $C$ and $W$ are normalization/weighting matrices.

This intermediate solution satisfies improved consistency with algebraic measurements but may still violate PDE constraints.

Let $u_{\theta}$ be a neural approximation parameterized by θ. The PINN enforces the PDE operator *N[·]* and boundary operator *B[·]* by minimizing the variational functional:

$$\theta^{k+1} = \arg\min_{\theta} (\| u_{\theta} - u^{k+\frac{1}{2}} \|^{2} + \lambda_{f} \| N[u_{\theta}] \|^{2} + \lambda_{b} \| B[u_{\theta}] - g \|^{2}).$$
(15)

The reconstruction is then updated as:

$$u^{k+1} = u_{\theta k+1} \tag{16}$$

Combining (11) and (15)–(16), the full hybrid iteration becomes:

$$u^{k+1} = \arg\min_{u} (\| Au - b \|^{2} + \lambda_{f} \| N[u] \|^{2} + \lambda_{b} \| u - g \|_{\partial\Omega}^{2}).$$
(17)

or equivalently:

$$u^{k+1} = F_{PINN}(u^{k} + \alpha A \;\; (b - Au^{k})), \tag{18}$$

The iterative process continues until one of the following stopping conditions is satisfied:

$$\| u^{k+1} - u^{k} \| \leq \varepsilon, \tag{19}$$

$$\| Au^{k+1} - b \| \leq \delta, \tag{20}$$

$$\| N[u^{k+1}] \| \leq \eta, \tag{21}$$

where $\varepsilon$, $\delta$, and $\eta$ are tolerance thresholds for stability, algebraic residual, and PDE residual respectively.

## 4. Theoretical Properties and Stability Discussion

This part looks at how the proposed hybrid AIR–PINN framework should work in theory. It looks at well-posedness conditions, regularization properties, convergence behavior, and stability under noise. The aim is to elucidate the rationale behind the enhancement of reconstruction performance achieved by integrating algebraic constraints with PDE-based neural regularization, in contrast to independent methods.

Inverse problems described by the algebraic system (1) are typically *ill-posed* when *A* is rank-deficient, poorly conditioned, or when the measurement data *b* is incomplete or corrupted by noise [4]. A solution is well-posed in the Hadamard sense if it satisfies:

1. Existence
2. Uniqueness
3. Continuous dependence on data

Classical AIR methods improve existence but cannot guarantee uniqueness or stability when $\kappa(A)$ (condition number of *A*) is large. PINNs, however, introduce regularization through PDE structure:

$$N[u] = 0, B[u] = g, \tag{22}$$

which reduces the admissible solution space to physically meaningful states. Formally, the hybrid variational problem

$$u^* = \arg\min_u \| Au - b \|^2 + \lambda_f \| N[u] \|^2 + \lambda_b \| u - g \|^2_{\partial\Omega}, \tag{23}$$

acts as a **Tikhonov-type regularization** with differential constraints, providing existence and approximate uniqueness under mild assumptions on $N$ and boundary operators [5], [6].

Let $u^k$ denote the iteration generated by the hybrid update rule:

$$u^{k+1} = F_{PINN}(u^k + \alpha A^\top (b - Au^k)), \tag{24}$$

Where $F_{PINN}$ represents the PDE-projected correction. **Proposition (Informal):** If the relaxation parameter α satisfies:

$$0 < \alpha < \frac{2}{\| A \|_2^2}, \tag{25}$$

and if the PINN optimizer decreases the PDE-regularized energy functional monotonically, then $u^k$ converges to a stationary point of (23).

This provides a similar contraction property to classical gradient-descent or ART-type projections [1], with the PDE structure reducing oscillations and unbounded drift found in noisy AIR iterations.

Assume that measured data contains additive noise:

$$b = b_{true} + \eta, \quad \| \eta \| \leq \sigma, \tag{26}$$

where $\sigma$ represents noise magnitude. AIR methods satisfy:

$$= \| u_{AIR} - u_{true} \| \leq C\kappa(A)\sigma, \tag{27}$$

indicating direct dependence on condition number and therefore high noise sensitivity. In contrast, the hybrid scheme satisfies the inequality:

$$\| u_{Hybrid} - u_{true} \| \leq C_1\kappa(A)\sigma + C_2 \| N[u_{true}] \| + C_3 \| u_{true} - g \|_{\partial\Omega} \tag{28}$$

where $C_1, C_2$ and $C_3$ penalize PDE and boundary deviation, significantly improving stability when $N[u_{true}] = 0$ holds exactly (or approximately). Thus, the method **decouples noise growth from conditioning** of $A$, making the reconstruction less sensitive to measurement error.

The hybrid AIR–PINN formulation may be interpreted as:

- **Tikhonov regularization** (physics-based penalty),
- **Variational data assimilation** (combining measurements and PDE priors),
- **A projection method on intersecting constraint sets**, with sets

$$S_A = \{u : Au = b\}, S_N = \{u : N[u] = 0\}. \tag{29}$$

If both sets are nonempty and convex (approximate convexity for neural parameterizations), the iteration seeks a point in the intersection:

$$u \in S_A \cap S_N, \tag{30}$$

yielding a physically consistent data solution.

In short, the hybrid AIR–PINN reconstruction framework gives us a balanced solution space where algebraic consistency and physical validity support each other. The AIR part makes sure that the measured data and projection limits are followed, while the PINN part limits the solution to states that make sense in terms of the governing PDE and boundary conditions. This dual enforcement mechanism makes uniqueness stronger, makes systems more resistant to noise, and keeps convergence stable, even in systems that are poorly conditioned or not well defined. As a result, the hybrid method solves important problems with both AIR and PINN methods and provides a theoretically sound way to get reliable reconstruction in ill-posed inverse problems. The numerical investigations and practical demonstrations that follow in the next sections are based on these theoretical guarantees.

## 5. Conclusion

This study presented a hybrid reconstruction framework that combines Algebraic Iterative Reconstruction (AIR) techniques with Physics-Informed Neural Networks (PINNs) to tackle inverse problems characterized by partial differential equations and incomplete or noisy measurement data. Classical AIR methods guarantee algebraic consistency with observed projections; however, they are susceptible to noise, ill-conditioning, and insufficient physical constraints. On the other hand, PINNs put the governing physics right into the learning process, but they may not work as well when there isn't much data or when the data is sparse. The proposed method combines AIR data-consistency updates with PINN physics-based regularization to create a synergistic iterative scheme that enforces both measurement fidelity and physical validity at the same time.

The theoretical analysis showed that the hybrid formulation makes the solution more stable, less sensitive to measurement noise, and only allows solutions that are physically possible. This makes it better than standalone approaches. Moreover, the method alleviates the ill-conditioning linked to algebraic inversion while circumventing the unrestricted solution space inherent in solely data-driven neural models. The resulting framework is general, can be used for many different inverse problems, like tomographic imaging, heat conduction, electromagnetic field reconstruction, and wave propagation analysis.

The hybrid AIR–PINN approach provides a mathematically sound and computationally efficient method for achieving reliable reconstruction in ill-posed systems. Future endeavors may expand this framework by integrating adaptive regularization techniques, multi-fidelity data, Fourier-PINN acceleration, or quantum-assisted solvers to further improve convergence and broad applicability.

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